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HAN-SHOU LIU

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OF THE PLANET MERCURY

Han-Shou Liu
Geodynamics Branch
Trajectory Analysis and Geodynamics Division

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Goddard Space Flight Center
Greenbelt, Maryland

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ABSTRACT

The variation in solar heating due to the resonance rotation of Mercury produces periodic elastic deformations on the surface of the planet. The thermal stress and strain fields under Mercury's surface are calculated after certain simplifications. It is shown that deformations penetrate to a greater depth than the variation of solar heating, and that the thermal strain on the surface of the planet pulsates with an amplitude of 4×10^{-3} and a period of 176 days.

THERMAL ELASTIC DEFORMATIONS OF THE PLANET MERCURY

INTRODUCTION

In this paper, we investigate the thermal elastic deformations of the planet Mercury by means of equations of equilibrium and compatibility in which the variation of solar heating due to resonance rotation has been involved. In view of the localized nature of the maximum thermal stress and strain induced on the surface of Mercury the half-space model is adopted for the analysis.

TEMPERATURE FIELD

We consider a half-space in the xyz coordinate system with a plane boundary. Let a temperature wave T with cyclical frequency ω and amplitude T_0 be propagated on the plane $x = 0$ with velocity v in the direction of the y axis. Then the temperature wave T is described by

$$T = T_0 \cos \omega \left(t - \frac{y}{v} \right) \quad (1)$$

Because of the orientation of Mercury relative to the sun during each perihelion passage (Liu, 1971) the variation of solar heating on the surface of the planet can be taken into account by the introduction of a factor $\cos(\omega/2v)y$. The temperature distribution on the surface of Mercury is then given by the expression

$$T_{(0,y,t)} = T_0 \cos(\omega t - 2\epsilon y) \cos \epsilon y \quad (2)$$

where $\omega = 4.2 \times 10^{-7} \text{ sec}^{-1}$, $\epsilon = \omega/2v = 2 \times 10^{-9} \text{ cm}^{-1}$.

The problem of determining the temperature field $T(x,y,t)$ is to find the bounded solution of the heat-conduction equation

$$\rho c \frac{\partial T_{(x,y,t)}}{\partial t} = \frac{\partial}{\partial x} \left[k \frac{\partial T_{(x,y,t)}}{\partial x} \right] + \frac{\partial}{\partial y} \left[k \frac{\partial T_{(x,y,t)}}{\partial y} \right] \quad (3)$$

for

$$0 < x < \infty$$

$$-\infty < y < \infty$$

$$-\infty < t$$

In equation (3) ρ is density, c the specific heat and k the thermal conductivity. If we assume that Fourier's law holds for the propagation of the temperature wave, the temperature field is

$$T_{(x,y,t)} = T_0 e^{-\delta x} \cos(\omega t - \delta x - 2\epsilon y) \cos \epsilon y \quad (4)$$

where $\delta = \sqrt{(\omega \rho c)/(2k)}$.

THERMAL STRESS

On Mercury's surface a progressing temperature wave $T(o,y,t)$ is given in a way to account for the variation in solar insolation; it generates the temperature field (4) along the equator of the planet. From this temperature field, we can find the thermal stress. The components of thermal stress are related to a stress function Φ by (Landau and Lifshitz, 1959)

$$\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2}$$

$$\sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} \quad (5)$$

$$\sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2}$$

The stress components must satisfy the equilibrium conditions

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \quad (6)$$

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0$$

and the condition of compatibility

$$\left(\frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \right) \Phi + \frac{\alpha E}{1 - \nu} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) T_{(x,y,t)} = 0 \quad (7)$$

where E is Young's modulus, ν Poisson's ratio, and α the coefficient of linear thermal expansion. It is noted that the external forces in equation (6) are omitted because the variation of body forces is small and simple calculations show that their influence is negligible for the case of the planet Mercury.

Let $\Phi = \phi - \psi$. Then equation (7) is equivalent to a system of two differential equations

$$\left(\frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \right) \phi = 0 \quad (8)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi = \frac{\alpha E}{1 - \nu} T_{(x,y,t)} \quad (9)$$

Since the solutions of equation (9) can be included in solutions to the biharmonic equation (8), only a particular integral is needed for the solution of equation (8). In addition, the boundary conditions are

$$\sigma_{xx} \big|_{x=0} = 0 \text{ and } \sigma_{xy} \big|_{x=0} = 0 \quad (10)$$

because there is no vertical and shear stress on the surface $x = 0$. Thus the thermal stress problem is reduced to solving equations (8) and (9) with conditions in equation (10). The stress will then be given by equation (5) for $\Phi = \phi - \psi$.

The periodicity of the temperature variation at the boundary leads to a periodic thermal stress under the surface. If we use the local character of the solar heating during perihelion passage, the stress function can be expressed in the form

$$\begin{aligned}\phi &= k_1 f_1(x) \exp(-i\epsilon y) \\ \psi &= k_2 f_2(x) \exp(-i\epsilon y)\end{aligned}\tag{11}$$

By substituting equation (11) into equations (8) and (9), we obtain ordinary differential equations. The unknown coefficients can be determined from the boundary conditions in equation (10) by using equation (5). The results of the components of thermal stress are

$$\begin{aligned}\sigma_{xx} &= \frac{\alpha E T_0}{1-\nu} \left\{ \left[\frac{\sqrt{2}\epsilon^2}{2\delta} x \cos\left(\omega t - \frac{\pi}{4}\right) - \frac{\epsilon^2}{2\delta^2} (1 + \epsilon x) \sin\omega t \right] e^{-\epsilon x} \right. \\ &\quad \left. + \frac{\epsilon^2}{2\delta^2} e^{-\delta x} \sin(\omega t - \delta x) \right\} \cos\epsilon y\end{aligned}\tag{12}$$

$$\begin{aligned}\sigma_{xy} &= \frac{\alpha E T_0}{1-\nu} \left\{ \left[\frac{\sqrt{2}\epsilon}{2\delta} (\epsilon x - 1) \cos\left(\omega t - \frac{\pi}{4}\right) - \frac{\epsilon^3}{2\delta^2} x \sin\omega t \right] e^{-\epsilon x} \right. \\ &\quad \left. + \frac{\sqrt{2}\epsilon}{2\delta} e^{-\delta x} \cos\left(\omega t - \delta x - \frac{\pi}{4}\right) \right\} \sin\epsilon y\end{aligned}\tag{13}$$

$$\begin{aligned}\sigma_{yy} &= \frac{\alpha E T_0}{1-\nu} \left\{ \left[\frac{\sqrt{2}\epsilon}{\delta} \left(1 - \frac{\epsilon}{2} x\right) \cos\left(\omega t - \frac{\pi}{4}\right) - \frac{\epsilon^2}{2\delta^2} (1 - \epsilon x) \sin\omega t \right] e^{-\epsilon x} \right. \\ &\quad \left. - e^{-\delta x} \cos(\omega t - \delta x) \right\} \cos\epsilon y\end{aligned}\tag{14}$$

These components of stress satisfy the equilibrium conditions as can be verified by substituting equations (12), (13) and (14) in equation (6).

INCLINATION AND DISPLACEMENT

The thermal strain (ϵ_{xx} , ϵ_{yy}) and displacement (U,V) are related to thermal stress (σ_{xx} , σ_{yy}) by means of the generalized Hooke's law

$$\epsilon_{xx} = \frac{\partial U}{\partial x} = \frac{1}{E} (\sigma_{xx} - \nu \sigma_{yy}) + \alpha T_{(x,y,t)} \quad (15)$$

$$\epsilon_{yy} = \frac{\partial V}{\partial y} = \frac{1}{E} (\sigma_{yy} - \nu \sigma_{xx}) + \alpha T_{(x,y,t)} \quad (16)$$

Therefore, the inclination (θ) due to thermal elastic deformation can be calculated by

$$\begin{aligned} \theta &= \frac{\partial U}{\partial y} = \frac{\partial}{\partial y} \int \frac{\partial U}{\partial x} dx \\ &= \frac{\partial}{\partial y} \int \left[\frac{1}{E} (\sigma_{xx} - \nu \sigma_{yy}) + \alpha T_{(x,y,t)} \right] dx \end{aligned} \quad (17)$$

and the component of displacement V is

$$V = \int \left[\frac{1}{E} (\sigma_{yy} - \nu \sigma_{xx}) + \alpha T_{(x,y,t)} \right] dy \quad (18)$$

We substitute the stress components (12) and (14) in equation (17) and (18). The desired components of the deformation are

$$\begin{aligned} \theta &= \frac{\alpha T_0}{1 - \nu} \left\{ \frac{\sqrt{2}\epsilon}{2\delta} [(\nu + 1)(\epsilon x + 1) - 2\nu] e^{-\epsilon x} \cos\left(\omega t - \frac{\pi}{4}\right) \right. \\ &\quad - \frac{\epsilon^2}{2\delta^2} (\nu + 1)(\epsilon x + 2) e^{-\epsilon x} \sin \omega t + \frac{\sqrt{2}\epsilon}{2\delta} e^{-\delta x} \sin\left(\omega t - \delta x + \frac{\pi}{4}\right) \\ &\quad \left. - \frac{\sqrt{2}\epsilon^3}{2\delta^3} e^{-\delta x} \cos\left(\omega t - \delta x + \frac{\pi}{4}\right) \right\} \sin \epsilon y \end{aligned} \quad (19)$$

$$\begin{aligned}
V = & \frac{\alpha T_0}{1-\nu} \left\{ \frac{\sqrt{2}}{2\delta} [2 - (\nu + 1) \epsilon x] e^{-\epsilon x} \cos\left(\omega t - \frac{\pi}{4}\right) \right. \\
& - \frac{\epsilon}{2\delta^2} [\nu + 1 + (\nu - 1) \epsilon x] e^{-\epsilon x} \sin\omega t \\
& - \nu \frac{\sqrt{2}}{2\epsilon} \left(1 + \frac{\epsilon}{2\delta}\right) e^{-\delta x} \cos\left(\omega t - \delta x + \frac{\pi}{4}\right) \\
& \left. - \nu \frac{\sqrt{2}}{2\epsilon} \left(1 - \frac{\epsilon}{2\delta}\right) e^{-\delta x} \sin\left(\omega t - \delta x + \frac{\pi}{4}\right) \right\} \sin\epsilon y
\end{aligned} \tag{20}$$

LATITUDINAL STRESS AND STRAIN

The thermal stress and strain are accompanied by a stress component σ_{zz} given by

$$\sigma_{zz} = \nu (\sigma_{xx} + \sigma_{yy}) - \alpha E T_{(x,y,t)} \tag{21}$$

and by a strain component ϵ_{zz} given by

$$\epsilon_{zz} = -\frac{\nu}{E} (\sigma_{xx} + \sigma_{yy}) + \alpha T_{(x,y,t)} \tag{22}$$

Therefore, the latitudinal stress component σ_{zz} and strain component ϵ_{zz} are

$$\begin{aligned}
\sigma_{zz} = & \frac{\nu \alpha E T_0}{1-\nu} \left\{ \left[\frac{2\epsilon}{\delta} \cos\left(\omega t - \frac{\pi}{4}\right) - \frac{\epsilon^2}{\delta^2} \sin\omega t \right] e^{-\epsilon x} \right. \\
& \left. - \left[\frac{1}{\nu} \cos(\omega t - \delta x) - \frac{\epsilon^2}{2\delta^2} \sin(\omega t - \delta x) \right] e^{-\delta x} \right\} \cos\epsilon y
\end{aligned} \tag{23}$$

$$\epsilon_{zz} = -\frac{\nu \alpha T_0}{1 - \nu} \left\{ \left[\frac{\sqrt{2}\epsilon}{\delta} \cos\left(\omega t - \frac{\pi}{4}\right) - \frac{\epsilon^2}{\delta^2} \sin\omega t \right] e^{-\epsilon x} - \left[\frac{1}{\nu} \cos(\omega t - \delta x) - \frac{\epsilon^2}{2\delta^2} \sin(\omega t - \delta x) \right] e^{-\delta x} \right\} \cos\epsilon y \quad (24)$$

NUMERICAL RESULTS

The maximum value of the thermal elastic inclination, θ , in equation (19) occurs at $\omega t = \pi/4$ for $x = 0$. This implies that the surface inclination reaches its maximum value when Mercury is at a position which is 7 days after perihelion passage. Numerical calculations were carried out for $E = 10^{11}$ dyne cm^{-2} , $T_0 = 3 \times 10^2$ deg, $k = 0.5 \times 10^{-2}$ cal cm^{-1} sec^{-1} deg^{-1} , $\rho = 5$ g cm^{-3} , $c = 0.2$ cal g^{-1} deg^{-1} , $\nu = 0.3$ and $\alpha = 10^{-5}$ deg^{-1} . The main results are: $\theta|_{\text{max}} = 2.6'' \times 10^{-4}$, $|\epsilon_{zz}|_{\text{max}} = 4 \times 10^{-3}$ and $|\sigma_{yy}|_{\text{max}} = 4 \times 10^8$ dyne cm^{-2} . It should be pointed out that calculations of equation (19) show that the thermal elastic inclination, θ , penetrates to a greater depth than the temperature variation in solar heating because it is not only produced by the absolute value of the temperature, but also by the temperature gradient.

CONCLUSION

The variation of solar heating on Mercury's surface due to resonance rotation generates periodic thermal elastic inclination and displacement. From the foregoing analysis, we may draw the following concluding remarks.

- (1) The thermal stress on the surface of Mercury is about 4×10^8 dyne per cm^2
- (2) Thermal elastic inclination penetrates to a greater depth than the temperature variation of solar heating. The maximum value of surface inclination occurs when Mercury is in a position which is 7 days after perihelion.
- (3) Equations (19) and (20) describe the thermal elastic response to the variation of solar heating. The latitudinal strain on the surface of Mercury's thermal bulges pulsates with an amplitude of 4×10^{-3} in a period of 176 days.

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